地域分析
第54巻第2号
2015年12月

A Note on core nonemptiness of a nontransferable utility game based on the standard microeconomic model*

Ryoji Okatani

Abstract

We provide a proof of core nonemptiness of the nontransferable utility (NTU) game in cooperative game theory based on the standard microeconomic model, which is introduced by standard microeconomics textbooks. For example, continuous utility functions and compactness and convexity for available consumption sets are often used in textbooks. We use such assumptions and construct an NTU game without the peculiar settings of cooperative game theory. Furthermore, we show that if the game is balanced, the weak core is nonempty similar to the general NTU game.

Keywords: NTU game; core nonemptiness
JEL Classification Numbers: C02; C71

Contents

1 Introduction 1
2 Model 2
3 Weak core 4
4 Balancedness 4
5 Proof 6
Reference 8

1 Introduction

We provide a proof of core nonemptiness of the nontransferable utility (NTU) game based on the standard microeconomic model, which is usually introduced

* This work was supported by Aichi Gakuin University Sangyo Kenkyujo grant.

in standard textbooks such as *Microeconomic Theory* by Mas-Colell, Whinston, and Green (1995). In the general NTU game, some unfamiliar assumptions, for example, comprehensiveness, cylindricality, and characteristic function, are used. We construct an NTU game without the peculiar settings of cooperative game theory. Furthermore, we show that if the game is balanced, the weak core is nonempty, similar to the general NTU game.

We assume that utility functions are continuous; and locally nonsatiated; and attain minimum utility at a boundary of the consumption set. Furthermore, we assume that available consumption sets, which are similar to budget sets, are compact and convex. These assumptions are often used in microeconomics.

In the present paper, the only assumption that we use from cooperative game theory is balancedness. In the general NTU game, balancedness of the game is assumed for the characteristic function. We assume it for the available consumption sets since we consider balancedness as a rule for the allocation of goods.

The theorem of core nonemptiness is provided by Scarf (1967). Shapley and Vohra (1991) provide a simple proof of this theorem, and our proof is based on their proof. However, since our assumption of the balancedness of the game is based on the available consumption sets whose dimension is the number of goods, our proof becomes slightly more complicated.

We use the following mathematical notations. For $X \subset \mathbb{R}^\ell$, $\partial^\ell X$ is the set of all boundary points of X in $\mathbb{R}^\ell$, and $\mathrm{int}^\ell X$ is the set of all interior points of X in $\mathbb{R}^\ell$. For $x \in \mathbb{R}^\ell$ and $\varepsilon \in \mathbb{R}_{++}$, $B^\ell_\varepsilon(x) \subset \mathbb{R}^\ell$ is the open ball with center x and radius ε. For $Z = \prod_{i=1}^n X_i \subset \mathbb{R}^{n\ell}$, $\mathrm{int}^{n\ell} Z = \prod_{i=1}^n \mathrm{ri}^{n\ell} X_i = \prod_{i=1}^n \mathrm{int}^\ell X_i$ where $\mathrm{ri}^{n\ell} X_i$ is the relative interior of X_i.

2 Model

Let $N = \{1, \dots, n\}$ be the set of *players*, who each consume an ℓ-tuple bundle of goods. Let $L = \{1, \dots, \ell\}$ denote indexes of goods, and let $\mathbb{R}^\ell_+$ be the consumption set for each player. A generic element of player i's consumption set is denoted by $x_i \in \mathbb{R}^\ell_+$. Suppose that $\ell \geq n$. We call $x = (x_1, \dots, x_n) \in \mathbb{R}^{\ell n}_+$ an *allocation*. Players can form a *coalition*, which is denoted by an n-tuple vector $c \in \mathscr{N} := \{0, 1\}^n$. For example, the *grand coalition*, which is formed by all players, is denoted by $(1, 1, \dots, 1)$, and the coalition formed only by player 1 is denoted by $(1, 0, \ldots, 0)$. That is, if player i is in coalition c, then the value of the ith coordinate of c is 1, and if not, the value is 0. Let $\mathbf{1} = (1, 1, \ldots, 1) \in \mathscr{N}$ be the grand coalition. Let $M\colon \mathscr{N} \to \mathscr{P}(N)$ be another expression of the coalitions. For example, $M(\mathbf{1}) = N$, $M((1, 0, \ldots, 0)) = \{1\}$, $M((1, 1, 0, \ldots, 0)) = \{1, 2\}$, and so on. Let $f_i\colon \mathscr{N} \twoheadrightarrow \mathbb{R}^{\ell}_{+}, i = 1, \ldots, n$, be *player i's available consumption correspondence*. Player i can consume $x_i \in f_i(c)$ if coalition c is formed. Let $f\colon \mathscr{N} \twoheadrightarrow \mathbb{R}^{\ell n}_{+}$ be $f(c) = \prod_{i=1}^{n} f_i(c)$ for all $c \in \mathscr{N}$. We call $f_i(c)$ an *available consumption set*.

Available consumption sets are similar to budget sets when coalition c is formed. Thus, we suppose the next assumption.

Assumption 1. The available consumption correspondence f_i is convex and compact-valued for all $i \in N$.

Players can consume nothing, so free disposability for available consumption is assumed.

Assumption 2. For all players $i \in N$ and coalitions $c \in \mathscr{N}$, $\mathbf{0} \in f_i(c)$.

For all coalitions, a minimum positive consumption is assumed for all players.

Assumption 3. There exists $x_i \in \mathbb{R}^{\ell}_{++}$ such that $x_i \in f_i(c)$ for all $i \in N$ and $c \in \mathscr{N}$.

Next, we introduce the utility function. Let $u_i\colon \mathbb{R}^{\ell}_{+} \to \mathbb{R}$ be the *utility function of player i*. The following is assumed for utility functions.

Assumption 4. For all players $i \in N$, utility functions u_i are continuous on $\mathbb{R}^{\ell}_{+}$, and locally nonsatiated in $\mathbb{R}^{\ell}_{++}$, and $u_i(\mathbf{0}) = 0$. Furthermore, for all $x_i \in \mathbb{R}^{\ell}_{+}$, if $x_i^j = 0$ for some $j \in L$, then $u_i(x_i) = 0$.

By assumptions 1, 2, and 4, we have the following fact.

Fact 1. For all players $i \in N$, and for all coalitions $c \in \mathscr{N}$, there exists a maximum utility $u_i(x_i^*(c))$ such that $x_i^*(c) \in \partial^{\ell} f_i(c)$

Proof. By assumption 4, the utility function u_i is continuous on $f_i(c)$. By assumption 1, $f_i(c)$ is a compact set. Thus, there exists $x_i^*(c) \in f_i(c)$ such that u_i attains its maximum at $x_i^*(c)$. Suppose, on the contrary, that $x_i^*(c) \in \mathrm{int}^{\ell} f_i(c)$. There exists $\varepsilon \in \mathbb{R}_{++}$ such that $B^{\ell}_{\varepsilon}(x_i^*(c)) \subset \mathrm{int}^{\ell} f_i(c)$. By assumption 2, u_i is locally nonsatiated. Therefore, there exists $y_i \in B^{\ell}_{\varepsilon}(x_i^*(c))$ such that $u_i(y_i) > u_i(x_i^*(c))$. This is a contradiction. Thus, we have $x_i^*(c) \in \partial^{\ell} f_i(c)$. □

We call $(N, (f_i)_{i \in N}, (u_i)_{i \in N})$ the *NTU game based on the standard microeconomic model*.

3 Weak core

Generally, core is the set of allocations obtained by the grand coalition, and the allocations are not dominated by any coalition. In this paper, we use strong domination as the definition of domination. An allocation $x=(x_1,\ldots,x_n)\in\mathbb{R}_+^{\ell n}$ is *strongly dominated* by an allocation $y=(y_1,\ldots,y_n)\in\mathbb{R}_+^{\ell n}$ via a coalition $c\in\mathscr{N}$ if

$$\exists y\in f(c), \forall i\in M(c), u_i(y_i)>u_i(x_i).$$

We can define the weak core. The set $X^*\subset f(\mathbf{1})$ is the *weak core* if no allocation $x\in X^*$ is strongly dominated by any coalition. That is,

$$X^*=\{x\in\mathbb{R}_+^{\ell n}: x\in f(\mathbf{1}) \text{ and } \neg(\exists c\in\mathscr{N}, \exists y\in f(c), \forall i\in M(c), u_i(y_i)>u_i(x_i))\}.$$

The next fact is useful.

Fact 2. The weak core has the following property.

$$X^*\subset f(\mathbf{1})\setminus\bigcup_{c\in\mathscr{N}}\operatorname{int}^{n\ell}f(c).$$

Proof. For all $x^*\in X^*$, we have $x^*\in f(\mathbf{1})$ by definition of X^*. Suppose, on the contrary, that $x^*\in\bigcup_{c\in\mathscr{N}}\operatorname{int}^{n\ell}f(c)$. Then, there exists $c\in\mathscr{N}$ such that $x^*\in\operatorname{int}^{n\ell}f(c)$. Then, there exist $\varepsilon_i\in\mathbb{R}_{++}$ such that $B_{\varepsilon_i}^{\ell}(x_i^*)\subset f_i(c)$ for all $i\in N$. Since u_i is locally nonsatiated, we have

$$\forall i\in N, \exists y_i\in B_{\varepsilon_i}(x_i^*)\subset f_i(c), u_i(y_i)>u_i(x_i^*).$$

Thus, x^* is dominated by y via c. This is a contradiction. □

4 Balancedness

The balancedness of games is a sufficient condition for the nonemptiness of X^*. First, we introduce the balanced collection, and then, we define the balanced game.

A set $\mathscr{B}\subset\mathscr{N}$ is a *balanced collection* if there exist non-negative weights $\lambda_c\in\mathbb{R}_+$ for $c\in\mathscr{B}$ such that

$$\sum_{c\in\mathscr{B}}\lambda_c c=\mathbf{1}.$$

The elements of balanced collection $\mathscr{B}$ are called *balanced sets*.

There is another expression of balancedness. For all $c \in N$, define

$$m^c = \frac{c}{|M(c)|}.$$

Let Δ^n be $n-1$-simplex such that $\Delta^n = \{g \in \mathbb{R}^n_+ : \sum_{i \in N} g_i = 1\}$. Clearly, $m^c \in \Delta^n$ for all $c \in \mathscr{N}$.

Fact 3. $\mathscr{B}$ is balanced if and only if $m^{\mathbf{1}} \in \mathrm{co}\{m^c : c \in \mathscr{B}\}$[1].

Proof. ($\Rightarrow$) Suppose that $\mathscr{B}$ is balanced collection, and $\mathscr{B} = \{c_1, \dots, c_k\}$. Then, there exist $\lambda_{c_j} \in \mathbb{R}_+, j = 1, \dots, n$ such that $\sum_{j=1}^k \lambda_{c_j} c_j = \mathbf{1}$. We have

$$\begin{aligned}
\sum_{j=1}^k \lambda_{c_j} c_j = \mathbf{1} \ \Leftrightarrow \ & \sum_{j=1}^k |M(c_j)| \lambda_{c_j} \frac{c_j}{|M(c_j)|} = \mathbf{1} \\
\Leftrightarrow \ & \sum_{j=1}^k |M(c_j)| \lambda_{c_j} m^{c_j} = \mathbf{1} \\
\Leftrightarrow \ & \sum_{j=1}^k \frac{|M(c_j)|}{|M(\mathbf{1})|} \lambda_{c_j} m^{c_j} = \frac{\mathbf{1}}{|M(\mathbf{1})|} \\
\Leftrightarrow \ & \sum_{j=1}^k \frac{|M(c_j)|}{|M(\mathbf{1})|} \lambda_{c_j} m^{c_j} = m^{\mathbf{1}}.
\end{aligned}$$

Thus, we show $\sum_{j=1}^k \frac{|M(c_j)|}{|M(\mathbf{1})|} \lambda_{c_j} = 1$. For all $i \in N$, define $\mathscr{B}_i$ as the set of $c \in \mathscr{B}$ such that $i \in M(c)$. Since $\sum_{c \in \mathscr{B}} \lambda_c c = \mathbf{1}$, we have $\sum_{c \in \mathscr{B}_i} \lambda_c = 1$. Thus, the following holds:

$$\sum_{j=1}^k |M(c_j)| \lambda_{c_j} = \sum_{i \in N} \sum_{c \in \mathscr{B}_i} \lambda_c = n = |M(\mathbf{1})|.$$

So, we have $\sum_{j=1}^k \frac{|M(c_j)|}{|M(\mathbf{1})|} \lambda_{c_j} = 1$; therefore $m^{\mathbf{1}} = \mathrm{co}\{m^c : c \in \mathscr{B}\}$

($\Leftarrow$) Suppose that $m^{\mathbf{1}} = \mathrm{co}\{m^c : c \in \mathscr{B}\}$. Then, there exists $\mu_c \in \mathbb{R}_{++}, c \in \mathscr{B}$ such that $\sum_{c \in \mathscr{B}} \mu_c = 1$ and $m^{\mathbf{1}} = \sum_{c \in \mathscr{B}} \mu_c m^c$. The following holds:

$$\begin{aligned}
m^{\mathbf{1}} = \sum_{c \in \mathscr{B}} \mu_c m^c \ \Leftrightarrow \ & \frac{\mathbf{1}}{|M(\mathbf{1})|} = \sum_{c \in \mathscr{B}} \mu_c \frac{c}{|M(c)|} \\
\Leftrightarrow \ & \mathbf{1} = \sum_{c \in \mathscr{B}} |M(\mathbf{1})| \mu_c \frac{c}{|M(c)|}.
\end{aligned}$$

Thus, let $\lambda_c := \frac{|M(\mathbf{1})|}{|M(c)|} \mu_c$, and we have $\mathbf{1} = \sum_{c \in \mathscr{B}} \lambda_c c$. □

Using balanced collections, we can define a balanced game. A game $(N, (f_i)_{i \in N}, (u_i)_{i \in N})$ is *balanced* if

1 For X, co$X$ is the convex hull of X.

$$\bigcap_{c\in\mathscr{B}} f(c) \subset f(\mathbf{1})$$

for all balanced collections $\mathscr{B}$.

The following theorem, proposed by Scarf (1967) is important in cooperative game theory, and is well known as Scarf's theorem in the general NTU game.

Theorem (Scarf (1967)) *If the game is balanced, the weak core is not empty.*

5 Proof

We prove the theorem in our standard microeconomic model. The proof is based on Shapley and Vohra (1991).

Lemma 1. Let a set Δ^ℓ be an $\ell-1$-simplex, *i.e.*, $\Delta^\ell=\{s\in\mathbb{R}^\ell_+ : \sum_{j\in L} s^j = 1,$ $s=(s^1,s^2,\ldots,s^\ell)\}$. There exists a continuous function such that $\varphi\colon \Delta^\ell \to \partial\bigcup_{c\in\mathscr{N}} f(c)$.

Proof. Let a function $\phi_i : \mathbb{R}_+\times\Delta^\ell \to \bigcup_{c\in\mathscr{N}} f_i(c)$ be $\phi_i(s)=t_i s$ for all $i\in N$. Then, ϕ_i is continuous. Let a correspondence $T_i\colon \Delta^\ell \twoheadrightarrow \mathbb{R}_+$ be

$$T_i(s)=\{t_i\in\mathbb{R}_+ \colon t_i s\in \bigcup_{c\in\mathscr{N}} f_i(c)\}.$$

Then, T_i is continuous and compact-valued. Let a function $\varphi\colon\Delta^\ell\to\bigcup_{c\in\mathscr{N}} f_i(c)$ be

$$\varphi_i(s)=\begin{cases}\max_{t_i\in T_i(s)}\phi_i(t_i,s) & \text{if } s\in\mathbb{R}^\ell_{++}\\ 0 & \text{if } s\in\mathbb{R}_+\setminus\mathbb{R}^\ell_{++}.\end{cases}$$

By Berge's maximum theorem, φ_i is continuous. Furthermore, we have $\varphi_i(s)\in\partial^\ell\bigcup_{c\in\mathscr{N}} f_i(c)$. □

Lemma 2. Let a correspondence $G\colon \Delta^\ell \twoheadrightarrow \Delta^n$ be

$$G(s)=\{m^c\in\Delta^n \colon c\in\mathscr{N} \text{ and } \varphi(s)\in f(c)\}.$$

Then G is a well-defined upper hemicontinuous correspondence.

Proof. Suppose that $s_\nu\to s^*$ as $\nu\to\infty, s_\nu\in\Delta^\ell$, $x_\nu\in G(s_\nu)$ for all $\nu\in\mathbb{N}$, and $x_\nu\to x^*$ as $\nu\to\infty$. Since $G(S)$ is finite and $x_\nu\to x^*$ as $\nu\to\infty$,

$$\exists\bar\nu\in\mathbb{N},\ \forall\nu\in\mathbb{N},\ \nu\geqq\bar\nu\ \Rightarrow\ x_\nu=x^*.$$

Thus,

$$\exists c\in\mathscr{N},\ m^c=x_\nu\ \forall\nu\geqq\bar\nu.$$

Therefore, $\varphi(s_\nu) \in f(c)$ for all $\nu \geqq \bar{\nu}$. Since φ is continuous and $f(c)$ is compact, we have $\varphi(s^*) \in f(c)$. That is, $x^* \in G(s^*)$ and G is upper hemicontinuous. □

Lemma 3. Let $\Delta^{\ell n}$ be an $\ell n - 1$-simplex. For $(s, g) \in \Delta^\ell \times \Delta^n$, let $\hat{s} \in \Delta^{\ell n}$ be $\hat{s}_i^j = \frac{s^j}{n}$ for all $i \in N$, and let $\hat{g} \in \Delta^{\ell n}$ be $\hat{g}_i^j = \hat{g}_k^j$. Let a function $h^j : \Delta^\ell \times \Delta^n \to \mathbb{R}_+$, $j \in L$ be

$$h^j(s, g) = \frac{\hat{s}_i^j + \max\left(\hat{g}_i^j - \dfrac{1}{\ell n}, 0\right)}{1 + \sum_{(j,i) \in L \times N} \max\left(\hat{g}_i^j - \dfrac{1}{\ell n}, 0\right)} n.$$

Let a function $h \colon \Delta^\ell \times \Delta^n \to \Delta^\ell$ be

$$h(s, g) = (h^1(s, g), \ldots, h^n(s, g)).$$

Furthermore, let a correspondence $H \colon \Delta^\ell \times \Delta^n \twoheadrightarrow \Delta^\ell \times \Delta^n$ be

$$H(s, g) = \{(h(s, g), \tilde{g}) \colon \tilde{g} \in \mathrm{co}(G(s))\}.$$

Then, H has a fixed point.

Proof. Since H is convex and compact-valued, H satisfies the conditions of Kakutani's fixed point theorem. □

Lemma 4. Let (s^*, g^*) be a fixed point of correspondence H. Then,

$$\mathscr{B}^* = \{c \in \mathscr{N} : \varphi(s^*) \in f(c)\}$$

is a balanced collection.

Proof. Let (s^*, g^*) be a fixed point of correspondence H, *i.e.*,

$$s^* = h(s^*, g^*) \text{ and } g^* \in \mathrm{co}(G(s^*)).$$

By the definition of H,

$$g^* \in \mathrm{co}\{m^c \colon c \in \mathscr{B}^*\}.$$

If $g^* = m^{\mathbf{1}}$, then $\mathscr{B}^*$ is balanced by Fact 3. Suppose that $g^* \neq m^{\mathbf{1}}$. Since (s^*, g^*) is a fixed point, we have

$$s^{*j} = \frac{s^{*j}_i + \max(g^{*j}_i - \frac{1}{\ell n}, 0)}{1 + \sum_{(j,i) \in L \times N} \max(g^{*j}_i - \frac{1}{\ell n}, 0)} n,$$

and then,

$$s^{*j}\left(\sum_{(j,i) \in L \times N} \max\left(g^{*j}_i - \frac{1}{n}, 0\right)\right) = n \max\left(g^{*j}_i - \frac{1}{\ell n}, 0\right)$$

for all $i \in N$. By $g^* \neq m^{\mathbf{1}}$ there exists $\tilde{i} \in N$ such that $g^*_{\tilde{i}} > \frac{1}{n}$. Thus, we have $\sum_{i \in N} \max(g^{*j}_i - \frac{1}{\ell n}, 0) > 0$. Let $J = \{j \in L: s^{*j} > 0\}$ and $K = \{k \in N: s^{*k} = 0\}$. If $K = \emptyset$, then $\max\left(g^{*j}_i - \frac{1}{\ell n}, 0\right) > 0$ for all $i \in N$. It is impossible. Thus, we have $K = \emptyset$. Hence, $s^* \in \mathbb{R}^\ell_+ \setminus \mathbb{R}^\ell_{++}$ and $\phi(s^*) = \mathbf{0}$. Since for all $c \in \mathscr{N}$, $\mathbf{0} \in f(c)$, we have $\mathscr{B}^* = \mathscr{N}$. Therefore, $m^{\mathbf{1}} \in \mathrm{co}\{m^c : c \in \mathscr{B}^*\}$, and $\mathscr{B}^*$ is balanced by Fact 3. □

Proof of the theorem. We show that the allocation derived by Lemma 4, $\varphi(s^*)$, is a core allocation. That is, $\varphi(s^*) \in X^*$. Suppose that $\varphi(s^*) \notin X^*$. By Lemma 4, $\varphi(s^*) \in f(c)$ for all $c \in \mathscr{B}^*$ and $\mathscr{B}^*$ is balanced. We have

$$\varphi(s^*) \in \bigcap_{c \in \mathscr{B}^*} f(c) \subset f(\mathbf{1}).$$

By Fact 2, we have

$$\varphi(s^*) \in \bigcup_{c \in \mathscr{N}} \mathrm{int}^{n\ell}(f(c)).$$

By a general property of interior,

$$\begin{aligned}\bigcup_{c \in \mathscr{N}} \mathrm{int}^{n\ell}(f(c)) &\subset \mathrm{int}^{n\ell}\left(\bigcup_{c \in \mathscr{N}} f(c)\right)\\ &= \mathrm{int}^{n\ell}\left(\prod_{i \in N} \bigcup_{c \in \mathscr{N}} f_i(c)\right)\\ &= \prod_{i \in N} \mathrm{int}^{\ell}\left(\bigcup_{c \in \mathscr{N}} f_i(c)\right).\end{aligned}$$

Thus, $\varphi_i(s^*) \in \mathrm{int}^\ell(\bigcup_{c \in \mathscr{N}} f_i(c))$. However, by the definition of φ, $\varphi_i(s^*) \in \partial^\ell \bigcup_{c \in \mathscr{N}} f_i(c)$ This is a contradiction. □

References

[1] Mas-Colell, A., M. Whinston, and J. Green (1995), *Microeconomic Theory*, Oxford University Press.

[2] Scarf, H. (1967), The core of an n person game, *Econometrica* 38: 50–69.

[3] Shapley, L. and R. Vohra (1991), On Kakutani's fixed point theorem, the K-K-M-S theorem and the core of a balanced game, *Economic Thoery* 1: 108–116.

受理日　平成 27 年 12 月 25 日