An elementary theory of a dynamic weighted digraph (2)

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Abstract

This paper is the second part of the revised version of my handout presented elsewhere (Chino, 2018a). We examine the trajectories of (1) one of our nonlinear complex difference equation models which describe changes in a dyadic relation between two objects over time, (2) random walk which was introduced by Pearson (1905), and (3) Brownian motion which was initiated by Brown (1827). We revisit some of the results in Chino (2018b) for these trajectories, first, and notice that some of the qualitative behaviors of our nonlinear dyadic system are reminiscent of random walk and Brownian motion. Then, we examine their mathematical properties further. We conclude that the three types of trajectories generated by different processes are different from each other in some respects.

Key words: complex difference equation, Hilbert space, Chino-Shiraiwa theorem, dynamic weighted digraph, chaos, random walk, Brownian motion.

1. Revisit of some of the results in Chino (2018b)

In Chino (2018b), we discussed a bit about our difference equation model described by a set of non-linear dyadic difference equations described below,

$$\begin{cases} z_{j,n+1} = z_{jn} + \alpha_{jk}^{(1)}(z_{jn} - z_{kn}) + \alpha_{jk}^{(2)}(z_{jn} - z_{kn})^2, \\ z_{k,n+1} = z_{kn} + \alpha_{kj}^{(1)}(z_{kn} - z_{jn}) + \alpha_{kj}^{(2)}(z_{kn} - z_{jn})^2, \end{cases} \quad (1)$$

which is denoted as Eq. (11) in that paper.

As discussed there, this type of system has a very desirable property in that we can utilize the heritage of the theory of the complex dynamical system developed in mathematics directly in classifying its trajectories. In fact, defining a new variable, $u_{jkn} = z_{jn} - z_{kn}$, and transforming it linearly, we have a new system

$$Z_{jk,n+1} = Z_{jk,n}^2 + \gamma_{jk}^{(1)}, \quad (2)$$

where $\gamma_{jk}^{(1)} = \frac{1}{2}\beta_{jk}^{(1)} - \frac{1}{4}\left[\beta_{jk}^{(1)}\right]^2$,

and $\beta_{jk}^{(1)} = 1 + \alpha_{jk}^{(1)} + \alpha_{kj}^{(1)}$.

The simplest system is such that $\gamma_{jk}^{(1)} = 0$ in Eq. (2). To attain this value, we set the α's as $\alpha_{jk}^{(1)} = 0.9564i$, $\alpha_{kj}^{(1)} = -1-0.9564i$, $\alpha_{jk}^{(2)} = 0.01$, and $\alpha_{kj}^{(2)} = 1.5375-0.9564i$.

It is well known that the Julia set of this system is the unit circle with origin 0 in C (equivalently, H), where C is the complex plane. Moreover, it is easy to show that this system has two fixed points, 1 and 0 (the origin), and these points are repelling and superattracting, respectively. <u>As a result, when we start from an arbitrary point on the unit circle except for the fixed point 1,</u>

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<u>the trajectory moves chaotically on the circle, at least theoretically.</u> (Notice that this circle is not a fractal since the unit circle is not self-similar.) As a trial, we set the initial value of this system as *i on the unit circle* in C, <u>since this point is not the fixed point of the system.</u> As pointed out in Chino (2018b), the strict Lyapunov exponent of this one-dimensional system, $Z_{jk,n+1} = Z_{jk,n}{}^2$, is $\lambda = \ln\ 2 = 0.69314718\cdots$, thus the system has chaos.

Next, we proceeded to examine qualitative behaviors of the original dyadic system described by Eq. (1). Fig. 1 shows trajectories of the four indices of the original dyadic system.

First, we computed the largest Lyapunov exponent of this dyadic system. Fig. 2 shows it. It converges to 0.693147148⋯, which is very close to that of the linearized system $Z_{jk,n+1} = Z_{jk,n}{}^2$, discussed above. This result clearly shows that the dyadic system has chaos.

Second, we shall zoom up trajectories shown in Fig. 1 to examine further the features of these trajectories. For example, Fig. 3 shows the expanded trajectories of the third time series in Fig. 1 from iteration 45,000 to 46,000. This trajectory is reminiscent of one-dimensional random walk or Brownian motion (Brown, 1828).

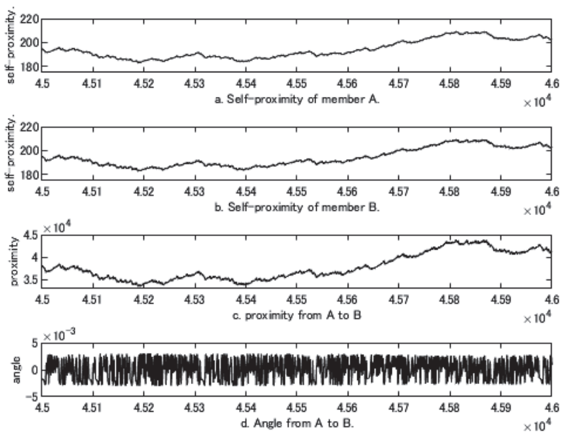

Fig. 1. Trajectories of the four indices of the original dyadic system. 10-a and 10-b are those of node j (member A) and node k (memberB), respectively. 10-c is the trajectory of the proximity from node j to node k. 10-d is the trajectory of the angle from node j to node k. (This figure is reproduced from Fig. 10 in Chino (2018b).

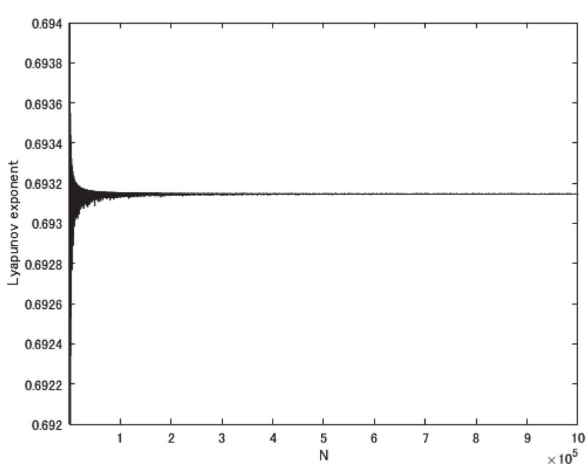

Fig. 2. The largest Lyapunov exponent of the original two dimensional system (This figure is reproduced from Fig. 11 in Chino (2018b).

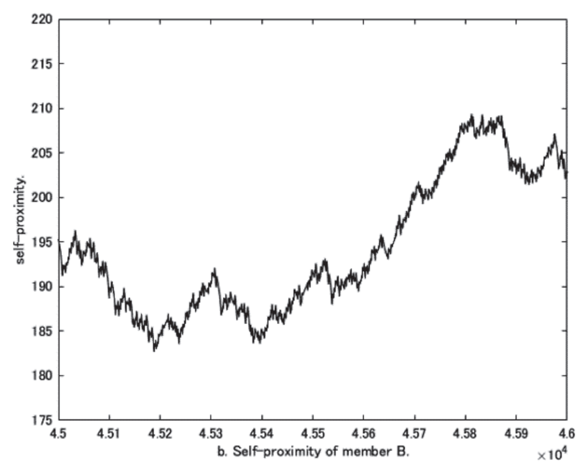

Fig. 3. Expanded trajectories of Fig. 1-c from iteration 45,000 to 46,000 (This figure is reproduced from Fig. 12 in Chino, 2018b).

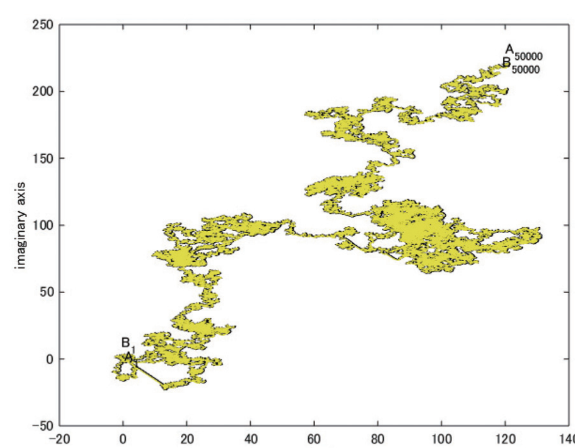

Fig. 4. Simultaneous plot of the two trajectories shown in Figs. 13 and 14 in Chino (2018b).

Third, we show trajectories of the dyadic system after 50,000 iterations. Fig. 4 shows the simultaneous plot of the trajectories of the two objects (nodes) using different colors, yellow and black.

Fig. 4 shows that the trajectories of two nodes are very close to each other. This point may be contrasted with those of the usual random work as well as the Brownian motion.

To compare the trajectories of our dyadic system with those of the random walk and the Brownian motion, we draw Figs. 5 and 6, which are trajectories of a simple random walk on the x-axis and a planar Brownian motion on the y-axis.

Figs. 7 and 8 are trajectories of a two-dimensional random walk and a planar Brownian motion after 20,000 iterations.

It is interesting to note that trajectories of our special dyadic system are reminiscent of the random walk or the Brownian motion.

In the following sections we abbreviate the fractal dimension as D_F, the Hausdorff-Besicovitch dimension as D_{HB}, the topological dimension as D_T, and the Euclidean dimension as D_E.

2. Mathematical properties of random walk

Random walk was introduced by Pearson (1905). A special random walk in the one-dimensional case is called the *simple random walk*. Assuming the n independent random variables, $Z_1, Z_2, \cdots, Z_n$, having values -1 or 1 (step size equals 1), the simple

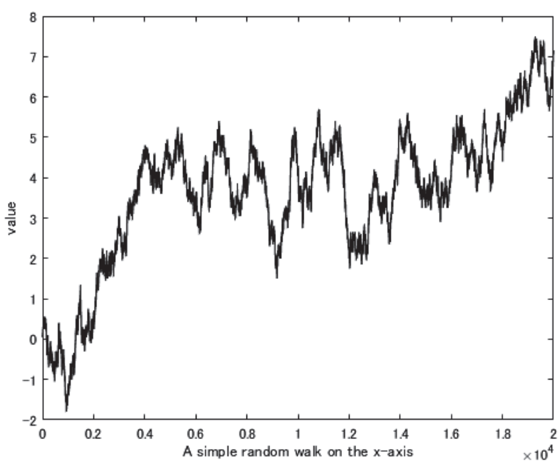

Fig. 5. Trajectories of a simple random walk on the x-axis.

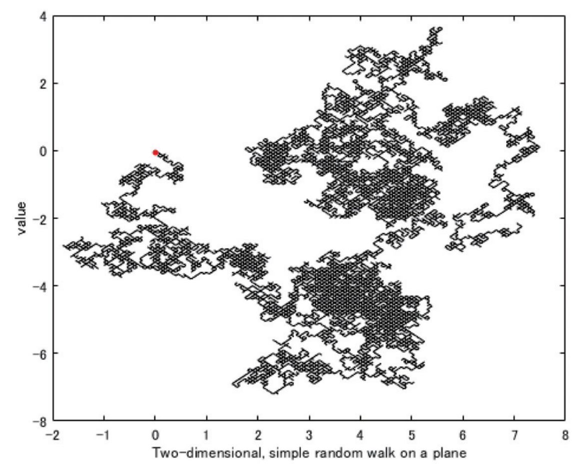

Fig. 7. Two-dimensional random walk after 20000 iterations.

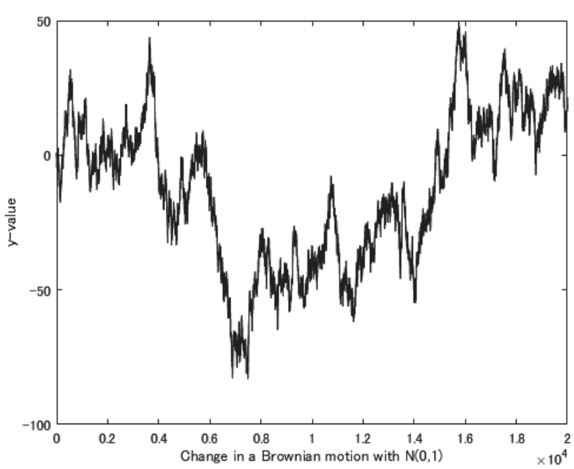

Fig. 6. Trajectory of a planar Brownian Motion in y-axis after 20000 iterations.

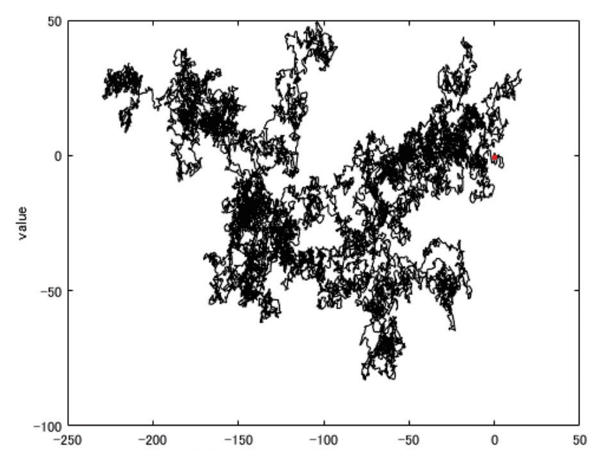

Fig. 8. Planar Brownian motion after 20000 iterations.

random walk on Z is defined as,

$$S_0 = 0, \quad S_n = \sum_{j=1}^{n} Z_j, \tag{3}$$

The random variable S_n has the following expectations: $\mathrm{E}(S_n) = 0$, and $\mathrm{E}(S_n^2) = n$.

If the step size approaches to zero in the simple random walk, we have the one-dimensional *Wiener process*, W_t, and the density function is written as

$$f_{W_t}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}, \quad \mathrm{E}(W_t) = 0, \text{and } \mathrm{V}(W_t) = t. \tag{4}$$

This process is equivalent to the physical Brownian motion in the one-dimensional case which will be discussed in the next section.

Random walk has no scale invariance, and thus is *not self-similar*. It is sometimes called a *statistically self-similar* (e.g., Mandelbrot, 1977). Moreover, in one-dimensional random walk (i.e., the simple random walk), both the fractal dimension (D_F) and the topological dimension (D_T) are 1's (De Gennes, 1979). Therefore, it is not a *fractal* (i.e., $D_F = D_T$). It is sometimes called the *discrete fractal*. In the two-dimensional random walk, D_F is 4/3 (de Gennes, 1979), and D_T is 1, thus it is a *fractal* (i.e., $D_F > D_T$). Since the variance of the random walk diverges, it is not a *chaos*.

3. Mathematical properties of Brownian motion

The so-called Brownian motion can be classified into two categories. One is the *physical Brownian motion* (e.g., Mandelbrot, 1977), and the other the *fractional Brownian motion (fBm)*. Physical Brownian motion is named after Brown (1827), while fBm was proposed by Mandelbrot and Van Ness (1968). According to them, the notion of fBm has already been considered elsewhere (Hunt, 1951; Kolmogorov, 1940; Lamperiti, 1962; Yaglom, 1958).

3.1 Physical Brownian motion

This is the random motion of tiny particles suspended in a fluid. Physically, it is thought of as a diffusion process, one-dimensional Brownian motion which Einstein (1905) described as

$$\mathrm{f(x, t)} = \frac{n}{\sqrt{4\pi D}} \frac{e^{-\frac{x^2}{4\nu t}}}{\sqrt{t}}. \tag{5}$$

Mathematically, physical Brownian motion is a *Gaussian process* as one of the stochastic process (e.g., Hansen, 2018), and historically it is also the Wi*ener process* introduced by Wiener (1923).

More generally, a process is *Gaussian* if all the finite dimensional marginal distributions(fidi) are Gaussian function, $\mathrm{t} \mapsto \mathrm{E}(X_t)$ $(s, t) \mapsto \mathrm{Cov}(X_s,\ X_t)$ as knowledge of these func tions allows us to write up every fidi. Here, the covariance matrices constructed from the covariance functions have for instance to be *symmetric* and *positive semi-definite (p.s.d.)* (e.g., Hunsen, 2018).

For example, the two-dimensional Gaussian process,

$$\begin{pmatrix} X_s \\ X_t \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} s & s \\ s & t \end{pmatrix}\right), \tag{6}$$

is a Brownian motion for any $0 \leqq s \leqq t$. If we consider the increment $X_t - X_s$ over the interval from *s* to *t*, we have

$$X_t - X_s \sim N(0, t - s). \tag{7}$$

Eq. (7) is equivalent to Eq. (5) which Einstein (1905) deduced.

In the one-dimensional physical Brownian motion, the Euclidian dimension (D_E) is 1, D_T is 1, and the Hausdorff-Besicovitch dimension (D_{HB}) (i.e., D_F) is 1.5 (Sprott, 2003). Therefore, the one-dimensional Physical Brownian motion is a *fractal* ($D_{HB} > D_T$). In the two-dimensional physical Brownian motion D_E is 2, D_T is 1, and D_{HB} is 2 (Mandelbrot, 1977). Therefore, the two-dimensional physical Brownian motion is not a *fractal* ($D_{HB} = D_T$). The physical Brownian motion is self-similar, and is sometimes called, *exactly self-similar* (e.g., Sprott, 2003). Moreover, it is not a chaos, because it is a diffusion process.

3.2 Fractional Brownian motion (fBm)

According to Khoshnevisian (2018), fBm is a generalization of the physical Brownian motion, and has a special covariance function,

$$\mathrm{E}(|X_t - X_s|^2) = |t-s|^{2\alpha}, \qquad s \leqq t \leqq 0. \quad (8)$$

Here, α is the *Hurst parameter* of X, and $\alpha > 0$. The *Hurst exponent H*, which is introduced in Mandelbrot (1977) is equivalent to the Hurst parameter α, in which H is defined as

$$R(d)/S(d) \propto d^H, \quad 0 \leqq \mathrm{H} \leqq 1, \quad (9)$$

where R(d) is the sample average river discharge between years 0 and d, while S(d) is a scaling factor. According to Mandelbrot (1977), D_{HB} is equal to the inverse of H, i.e., $D_{HB} = 1/\mathrm{H}$. If H is ½, then it is nothing but a (two-dimensional) *physical Brownian motion*, and thus. $D_{HB} = 2$.

Moreover, fBm is a special case of the Gaussian process. Falconer (2003) calls α the index of fBm. That is, he defines fBm of index-α ($0<\alpha<1$) as a Gaussian process X: $[0,\infty] \to \mathrm{R}$ on some probability space such that

(1) With probability 1, X(t) is continuous and X(0)=0;

(2) For every $t \geqq 0$ and $h>0$ the increment $X(t+h) - X(t)$ has the normal distribution with mean zero and variance $h^{2\alpha}$, so that

$$\mathrm{P}(X(t+h) - X(t) \leqq x) = (2\pi)^{-1/2} h^{-\alpha} \int_{-\infty}^{x} exp(-u^2/2h^{2\alpha})\, du. \quad (10)$$

4. Conclusion

In this paper we propose an elementary theory of a dynamical weighted digraph. In this theory we first assume that the weight matrix associated with a weighted digraph denotes the proximity strengths among nodes at any instant of time, and that it varies as time proceeds. Then we apply HFM (the Hermitian Form Model) (Chino & Shiraiwa, 1993) to the weight matrix, and obtain the configuration of objects (nodes) at any instant of time in a p-dimensional Hilbert space, H^p or an indefinite metric space. Here, we assume that the *Hermitian matrix* corresponding to the weight matrix is *positive-semidefinite*, in the most general case and its weights are measured at a ratio level. Then, we have the correspondence among digraph, weight matrix, and configuration of objects (nodes) at any instant of time. As a result, changes in digraphs over time are considered as changes in configurations of nodes in H^p over time.

Our elementary theory of dynamic digraph then assumes that these changes in the configurations of nodes can be described by a set of complex nonlinear difference equations in H^p in the most general case. The purpose of our theory is to classify elementary patterns of changes in digraphs over time, by assuming that there exists a latent process which governs these changes in digraph, which can be described by a set of complex nonlinear difference equations in H^p.

In this paper, we restrict the dimension p of the state space to one, and conducted some simulation studies in order to classify elementary patterns of changes in digraphs over time. It is easy to show that such patterns can be enumerated simply in the case when the latent dynamics are linear (Chino, 2017a, b; Chino, 2018a, b). Our major results in this paper are concerned with the patterns in the case when the latent dynamics have quadratic terms especially in the case where N=2 and q=2 in Eq. (1). As discussed elsewhere (Chino, 2017a, 2018a, 2018b), this type of system has a very desirable property in that we can utilize the heritage of the theory of the complex dynamical system developed in mathematics directly in classifying its trajectories.

In fact, the transformed new system of Eq. (1), i.e., Eq. (2) becomes the Mandelbrot set (Mandelbrot, 1977) in some cases. In the companion paper (Chino, 2018b), we have found that the original quadratic system Eq. (1) also exhibits *chaotic behaviors* which are very similar to random walks and Brownian motions. Therefore, in this paper we examine the similarities as well as differences among (1) random walks, (2) Brownian motions, and (3) our original quadratic system.

As regards the fractal property, one-dimensional (thus the simple) random walk has $D_F(D_{HB}) = 1$ and $D_T = 1$ (De Gennes,1979), and is not a fractal. However, the two-dimensional random walk has $D_F(D_{HB}) = 4/3$ and $D_T = 1$ (De

Gennes,1979), and is a fractal.

One-dimensional physical Brownian motion has $D_F(D_{HB})=1.5$ and $D_T=1$ (e.g., Sprott, 2003), and is a fractal. But, the two-dimensional physical Brownian motion (i.e., ordinary Brownian motion) has $D_F(D_{HB})=2$ and $D_T=2$ (Mandelbrot, 1977), and is not a fractal.

The one-dimensional fBm has $D_F(D_{HB})=2-\alpha$ and $D_T=1$ (Mandelbrot, 1977), and is a fractal (Falconer, 2003), where α is the index of fBm, or is also called the Hurst parameter (e.g., Khoshnevisian, 2018). It is sometimes called the random fractal (Sprott, 2003). In contrast, the two-dimensional fBm's has $D_F(D_{HB})=2$ and $D_T=2$ (Mandelbrot, 1977), and is not a fractal.

The boundary of the trajectory of the transformed one-dimensional complex system (i.e., Eq. (2)) of our quadratic system (i.e., Eq. (1)), which belongs to the Julia set in the unit circle with origin 0 in C (equivalently, one-dimensional Hilbert space), is not a fractal *geometrically*.

As for the self-similarity property, random walk has no scale invariance (Wikipedia, 2018) and thus is not self-similar. In fact, a computer simulation of simple random walk by Higuchi's method (Higuchi, 1988) shows that the log L(k) plotted against log k with base 2 is nonlinear in the whole range of abscissa.

Physical Brownian motion is self-similar (e.g., Sprott, 2003). Sometimes, it is called exactly self-similar.

Our quadratic system described by Eq. (2) might probably be self-similar. In fact, its computer simulation shows that all the four trajectories of this system shown in Fig. 1 are almost self-similar, since all the log L (k)-log k plots of these trajectories by Higuchi's method (Higuchi, 1988) show almost linear in the whole range of abscissa (e.g., Fig. 9, Fig. 10).

As for the chaos property, not only random walks but also Brownian motions (which include physical Brownian motions as well as fractal Brownian motions, i.e., fBm) might not be chaos, since all of these trajectories diverge as time proceeds. In contrast, the Lyapunov exponent (LE) of the trajectory of the transformed one-dimensional complex system of our original quadratic system computed numerically by a chaos software (i.e., Takahashi & Yamada, 2000) was very close to the theoretical Lyapunov exponents (i.e., ln 2). Moreover, the theoretical Lyapunov exponent of the original quadratic system was 0.693147148276891, which is equivalent to the theoretical LE of the transformed one-dimensional system up to seven decimal places. These results indicate that our original quadratic system has chaos.

Acknowledgements

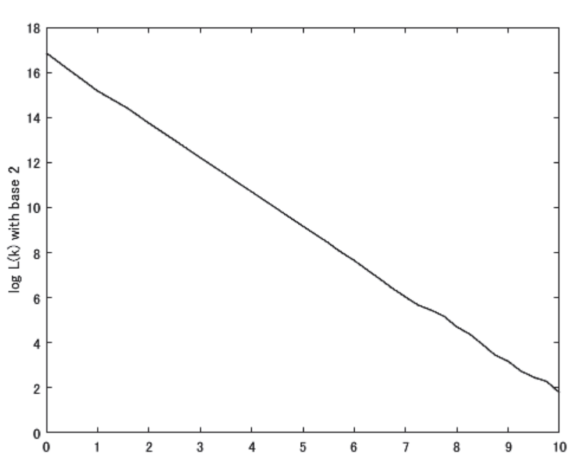

Fig. 9. The log L(k) against k plot of the trajectories of member A (node j) In Figure 1-a by Higuchi's method (Higuchi, 1988).

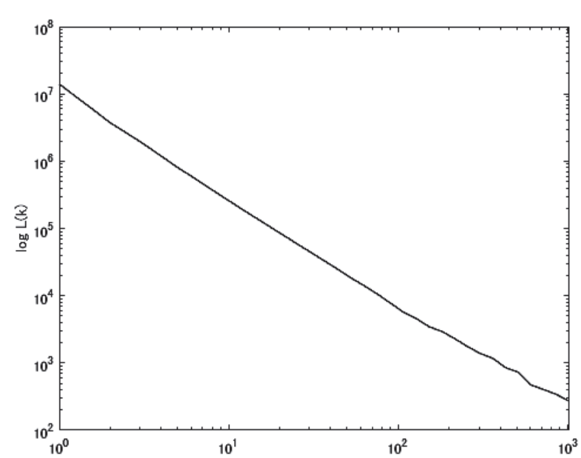

Fig. 10. The log L(k) against k plot of the trajectories of the proximity from member A(node j) to member B(node k) in Figure 1-c by Higuchi's method (Higuchi, 1988).

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